Math 279 Lecture 2 Notes

Daniel Raban

August 31, 2021

1 Stochastic Integration With Irregular Functions

1.1 Integration of rough deterministic functions

Our ultimate goal is to study stochastic PDEs, but before that, we need to study certain developments in studying stochastic ODEs from the 90s. For now, we are reviewing the stochastic differential equation

$$\frac{dx}{dt} = u(x,t) + \sigma(x,t)\xi(t),$$

where $\xi(t)$ is "white noise." As we discussed last time, we may make sense of this equation if we have a good candidate for

$$\int_0^t f(s)\,dg(s)$$

if f and g are as bad as Brownian motion. That is, we need to be able to deal with $f, g \in \mathcal{C}^\alpha$ for $\alpha < 1/2$. Last time, we learned that $h(t) = \int_0^t f\,dg = \lim_{n\to\infty} \sum_{j=0}^{2^n-1} f(s_j)(g(t_{j+1}) - g(t_j))$ with $s_j \in [t_j, t_{j+1}]$ and $t_j = t \cdot 2^{-n}$, provided that $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$ with $\alpha + \beta > 1$. Alternatively, we can state the following result of Young:

Theorem 1.1. *Given $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$ with $\alpha + \beta > 1$, there exists a unique $h \in \mathcal{C}^\beta$ such that $h(0) = 0$ and*

$$|h(t) - h(s) - f(s)(g(t) - g(s))| \leq [f]_\alpha [g]_\beta |t-s|^{\alpha+\beta}.$$

The idea is that we can approximate g by smooth functions to compute the integral, and if we keep doing this with better approximations, we will get the same answer, regardless of our choice of approximation.

An equivalent way to think about this is if $\mathscr{A} : \mathcal{C}^1 \times \mathcal{C}^1 \to \mathcal{C}^0$ by $\mathscr{A}(f, G) = fG'$, then this $\mathscr{A}$ has a continuous extension to $\widehat{\mathscr{A}} : \mathcal{C}^\alpha \times \mathcal{C}^\gamma \to \mathcal{C}^\gamma$ with $\alpha + \gamma > 0$. Here, $\gamma = \beta - 1$. This gives us a satisfactory candidate for fg', where $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$, $\alpha + \beta > 1$. The Radon-Nikodym theorem says that if a distribution is a measure, then we can multiply it by a function and we get another measure; this, by comparison says we can multiply a distribution (which can be worse than a measure) by a function as long as the function has enough regularity.

As we mentioned last time, Young's integral cannot be used for our equation. Imagine that we have $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$, and $\alpha + \beta \leq 1$ with $\alpha, \beta \in (0,1)$. What can be said about fg'? We may attempt to make sense of it by replacing g with a smooth approximation g_ε and examine $\lim_{\varepsilon \to 0} fg'_\varepsilon$. It turns out that the limit may not exist or the limit depends on the approximation.

In this context, let us examine the following question: Given Hölder f, g, consider the set $\mathscr{H}$ of h such that $h(0) = 0$ and for some C,

$$|h(t) - h(s) - f(s)(g(t) - g(s))| \leq C|t - s|^{\alpha+\beta}.$$

Observe that if $h, \widetilde{h} \in \mathscr{H}$, then $h - \widetilde{h} \in \mathcal{C}^{\alpha+\beta}$. In fact, given any $h^0 \in \mathscr{H}$,

$$\mathscr{H} = \{h^0 + k : k(0) = 0, k \in \mathcal{C}^{\alpha+\beta}\}.$$

Theorem 1.2 (Lyons-Victoire, 1999)**.** *$\mathscr{H} \neq \varnothing$ always.*

The multidimensional version of this theorem was proved by Martin Hairer in 2013 or so. In other words, if $f \in \mathcal{C}^\alpha(\mathbb{R}^d)$, $g \in \mathcal{C}^\beta(\mathbb{R}^d)$, then we have at least one candidate for "$f\nabla g$" (a function multiplied by a distribution). This is basically a distribution that near x, is "close" to $f(x)\nabla g$.

1.2 Integration of functions of Brownian motion

How does stochastic calculus fit into this framework? Let's go back to our original problem

$$\dot{x} = u(x,t) + \sigma(x,t)\xi, \qquad \xi = \dot{B}.$$

Our first attempt is to make sense of $\int_0^t F(B(s))\,dB(s)$.

It is not hard to show (using the strong law of large numbers) that

$$\lim_{n\to\infty} \sum_{j=0}^{2^{n-1}} [B(t_{j+1}) - B(t_j)]^2 = t$$

almost surely. Observe that

$$\int_0^t B\,dB \approx \begin{cases} \sum_i B(t_i)(B(t_{i+1}) - B(t_i)) & \text{Itô } (I) \\ \sum_i B(t_{i+1})(B(t_{i+1}) - B(t_i)) & \text{backward } (II) \\ \sum_i \frac{B(t_{i+1})+B(t_i)}{2}(B(t_{i+1}) - B(t_i)) & \text{Stratonovich } (III). \end{cases}$$

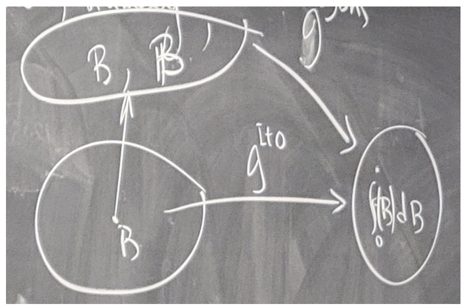
Observe that $II - I \to t$ as $n \to \infty$.

Itô's candidate was to define

$$\int_0^t F(B(s))\,dB(s) = \lim_{n\to\infty} \sum_i F(B(t_i))(B(t_{i+1}) - B(t_i)),$$

where the limit exists in $L^2(\mathbb{P})$. This is a fairly weak type of convergence, as opposed to Young's integral. Indeed, $B \mapsto \mathscr{I}(B) = \int_0^1 F(B(s))\,dB(s)$ is a only a measurable map and is *not* continuous. This is an unsatisfactory feature of Itô's theory.

Lyons made a very important observation, namely if we have a candidate for $\mathbb{B}(s,t) = \int_s^t (B(\theta) - B(s)) \otimes dB(\theta)$ (where the tensor denotes making a matrix out of this), then the map $(B, \mathbb{B}) \mapsto \mathscr{I}(B, \mathbb{B}) = \int_0^t F(B)\,dB$ is now continuous. (Though $\mathbb{B}(s,t)$ must satisfy some algebraic equations known as Chen's relations.)



For this theory, we can replace B with any function (or possibly random rough path) that is in $\mathcal{C}^\alpha$, provided that $\alpha > 1/3$.

1.3 The stochastic heat equation

We are now ready to discuss stochastic partial differential equations.

Example 1.1 (Stochastic heat equation)**.** The stochastic heat equation (SHE) is

$$u_t = \Delta u + \xi,$$

where ξ is white noise in (x,t). By this, we mean that ξ is a Gaussian process, $\mathbb{E}[\xi(x,t)] = 0$, and $\mathbb{E}[\xi(x,t)\xi(y,s)] = \delta_0(x-y, t-s)$ (to be formally defined later). One can show that $\xi \in \mathcal{C}^\alpha$ for any $\alpha < -d/2 - 1$. (Here, we are better off to use a "parabolic" metric, i.e. $|(x,t) - (y,s)|_{\mathrm{par}} = |x-y| + |t-s|^{1/2}$. Then Hölder means $\frac{|f(x,y) - f(y,s)|}{|(x,t)-(y,s)|_{\mathrm{par}}^\alpha}$.)

Because of "parabolic regularity" (which we will discuss later), we expect $u \in \mathcal{C}^{(-d/2+1)-}$. For example, when $d = 1$, $u \in \mathcal{C}^{1/2-}$ in the space variable, and it turns out that $u \in \mathcal{C}^{1/4-}$ in the time variable. In higher dimensions, this will not be a function; we have to live with distributions. We can make sense of this PDE by first using Duhamel to write

$$u(x,t) = \int p(x-y,t)u^0(y)\,dy + \int_0^t \int p(x-y,t-s)\underbrace{\xi(y,s)\,dy\,ds}_{W(dy,ds)},$$

where p is a fundamental solution of the heat equation and $W(dy, ds)$ is known as "cylindrical Brownian motion."